Ray and Eikonal Theory I

→ Rays, Eikonal Theory and Wave Propagation.

eikonal → icon
(Greek) ↓
image

OV:

→ here, seek to provide description of wave propagation in 'short wavelength' limit [N.B. How short? - see HW on parabolic wave equation].
→ relevant to semi-classical limit of QM
→ description is in terms of rays - paths followed by wave

Now:

- from HW, Fermat's minimum time principle (1662)

i.e. $T = \int_1^2 \frac{ds}{c(\underline{x})} = \frac{1}{c_0} \int_1^2 ds\, n(\underline{x})$

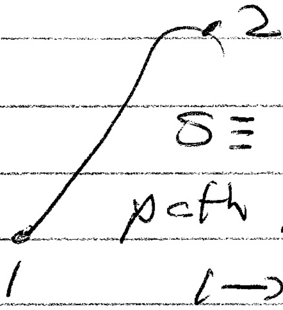
T: travel time; n: index; $ds\, n(\underline{x})$: 'ray Lagrangian'

$\delta T = 0 \Rightarrow$ ray path.

Generalizing the HW:

Fermat ⇒

$$0 = \delta \int_1^2 n(\underline{x}(\sigma))\, d\sigma$$



$\sigma \equiv$ ray path parameter $1 \to 2$

$$= \delta \int_1^2 n(\underline{x}(\sigma)) \left(\frac{d\underline{x}}{ds} \cdot \frac{d\underline{x}}{ds} \right)^{1/2} ds$$ (dummy time)

$$\equiv \text{[redacted]} \int_1^2 L\, ds$$

⇒

$$0 = \int_1^2 \left(\frac{\partial L}{\partial \underline{x}} \cdot \delta \underline{x} + \frac{\partial L}{\partial \left(\frac{d\underline{x}}{ds}\right)} \cdot \delta \left(\frac{d\underline{x}}{ds}\right) \right)$$

$$= \text{e.p.} + \int_1^2 \left(\frac{\partial L}{\partial \underline{x}} \cdot \delta \underline{x} - \frac{d}{ds} \left(\frac{\partial L}{\partial \left(\frac{d\underline{x}}{ds}\right)} \right) \delta \underline{x} \right)$$

⇒

$$\frac{\partial L}{\partial \underline{x}} - \frac{d}{ds} \left(\frac{\partial L}{\partial \left(\frac{d\underline{x}}{ds}\right)} \right) = 0$$

$$L = n(\underline{x}(s)) \left(\frac{d\underline{x}}{ds} \cdot \frac{d\underline{x}}{ds} \right)^{1/2}$$

Crank ⇒

1/2

if $|\dot{\underline{x}}| = \left[\frac{d\underline{x}}{ds} \cdot \frac{d\underline{x}}{ds}\right]$

$$\boxed{|\dot{\underline{x}}| \frac{\partial n}{\partial \underline{x}} - \frac{d}{ds}\left(n(\underline{x}) \frac{\dot{\underline{x}}}{|\dot{\underline{x}}|}\right) = 0}$$

→ general expression

→ $\partial n / \partial \underline{x}$ ⟺ effective force on ray
($U \leftrightarrow n$)

→ $n(\underline{x}) \frac{\dot{\underline{x}}}{|\dot{\underline{x}}|}$ ⟺ defines generalized momentum analogue.

$\left(n(\underline{x}) \frac{d\underline{x}}{ds}\right)$

Note: $ds^2 = d\underline{x} \cdot d\underline{x}$
so $|\dot{x}| = 1$

$$\Rightarrow \boxed{\frac{\partial n}{\partial \underline{x}} - \frac{d}{ds}\left(n(\underline{x}) \frac{d\underline{x}}{ds}\right) = 0}$$

is equivalent.

→ A bit of geometry:

$$\frac{d}{ds}\left(n(\underline{x})\frac{d\underline{x}}{ds}\right) - \frac{\partial n}{\partial \underline{x}} = 0 \qquad \to \text{ray equation}$$

⇒

$$n(\underline{x})\frac{d^2\underline{x}}{ds^2} + \left(\frac{\partial n}{\partial \underline{x}}\cdot\frac{d\underline{x}}{ds}\right)\frac{d\underline{x}}{ds} = \frac{\partial n}{\partial \underline{x}}(\underline{x})$$

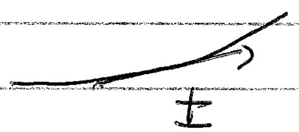
$$\boxed{\frac{d^2\underline{x}}{ds^2} = \frac{1}{n(\underline{x})}\frac{\partial n}{\partial \underline{x}} - \frac{1}{n(\underline{x})}\left(\frac{\partial n}{\partial \underline{x}}\cdot\frac{d\underline{x}}{ds}\right)}$$

What does it mean?

→ $d\underline{x}/ds$ is unit tangent to ray.

i.e. $ds\,ds = d\underline{x}\cdot d\underline{x}$

$\underline{t} = d\underline{x}/ds$



so

→ $d^2\underline{x}/ds^2$ corresponds to ray curvature K.

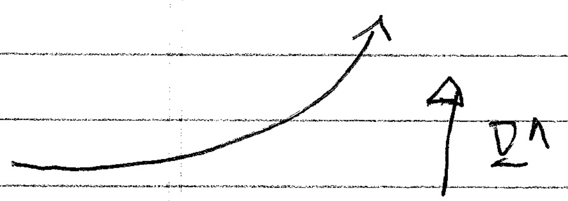
$$1/|\underline{K}| \equiv \text{effective radius of curvature}$$

so

$$\underline{K} = \frac{1}{n}\underline{\nabla} n - \frac{1}{n}(\underline{t}\cdot\underline{\nabla} n)\,\underline{t}$$

$$= \frac{1}{n}\left(\underline{\nabla} n \cdot \underset{\text{unit normal to path}}{\underbrace{\hat{\underline{n}}_0}}\right)\hat{\underline{n}}_0$$

Loosely put, ray curves toward region of increasing index.



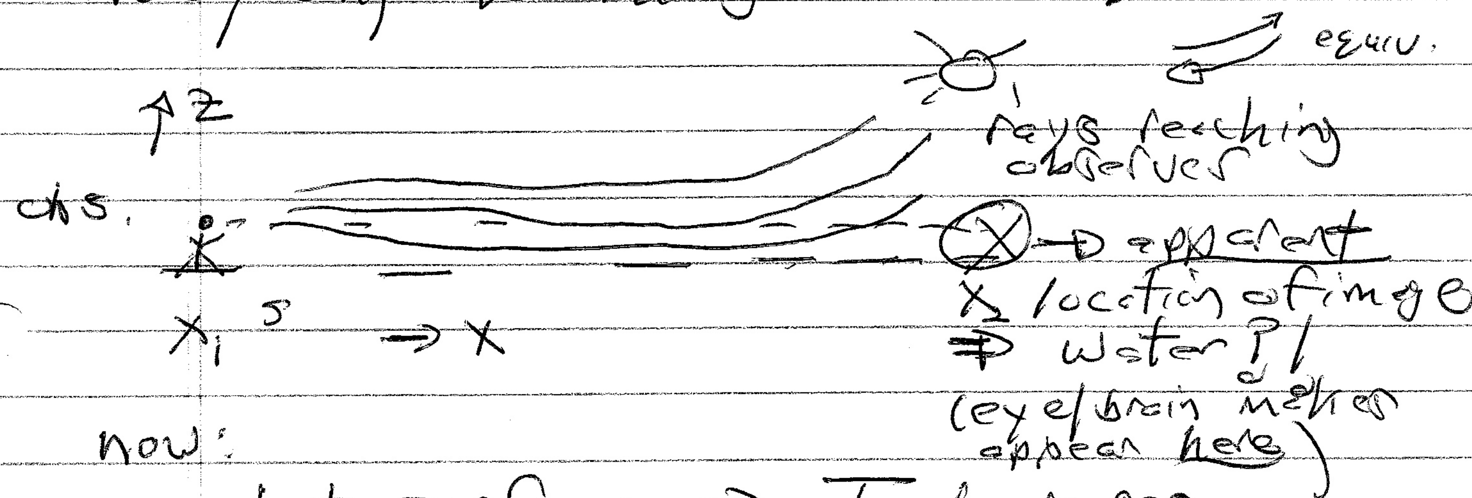
mirage ⟺ mirror

[redacted] 8 C

→ Mirages (see Wikipedia)

- mirages are optical illusions of reflection [redacted] from water etc. which occur in deserts, etc.

- how/why do mirages occur? [inferior mirage]



now:

- hot surface ⟹ T decreases, air density increases with height

- index $n \sim$ density.

- so, reasonable to take index $\sim z$

$$n(z) = n_0(1+\alpha z).$$

- Now, Fermat ⟹ ray from:

$$\delta\int\left(1+(dz/dx)^2\right)^{1/2} n(z) = 0$$

So

$$\frac{d}{dx}\left(\frac{n(z)}{\left(1+(dz/dx)^2\right)^{1/2}}\frac{dz}{dx}\right) = \left(1+\left(\frac{dz}{dx}\right)^2\right)^{1/2}\frac{dn}{dz}$$

$dz/dx \equiv \dot{z}$

$\Rightarrow$

$$\frac{d}{dx}\left(\frac{n_0(1+\alpha z)}{(1+\dot{z}^2)^{1/2}}\,\dot{z}\right) = n_0(1+\dot{z}^2)^{1/2}\,\alpha$$

For $\sim$ horizontal rays,

$\dot{z}^2 \ll 1$

$\alpha z \ll 1$

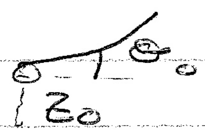
$\Rightarrow$

$$\frac{d^2z}{dx^2} \cong \alpha$$

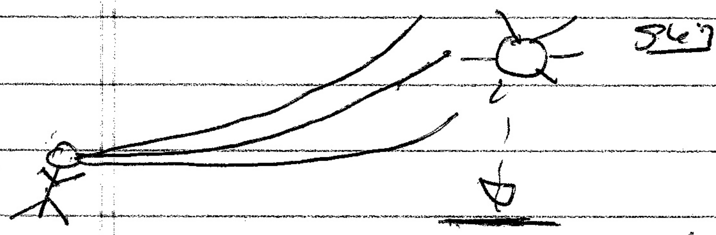
$\therefore$ then have:

$$z(x) = \left(\frac{\alpha}{2}x^2 + \tan\theta_0\, x + z_0\right)$$

inclination



then rays diverge parabolically,



apparent location
(shimmering, bright light)

⇒ mirage
(appears like reflection
from water)

Origin of shimmer?

Now, consider:

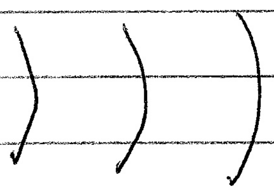
→ Helmholtz Eqn.

$$\nabla^2\psi + \frac{\omega^2}{c(x)^2}\psi = 0$$

↳ index

$$1/c(\underline{x})^2 \equiv \frac{n(\underline{x})^2}{c_0^2}$$ → ref. speed

→ consider phase front

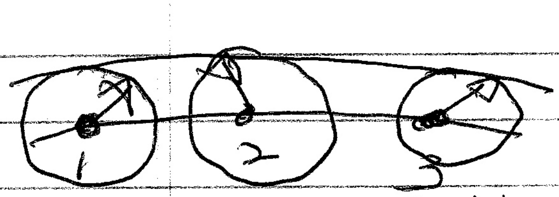


ϕ = const. surfaces

↓

phase

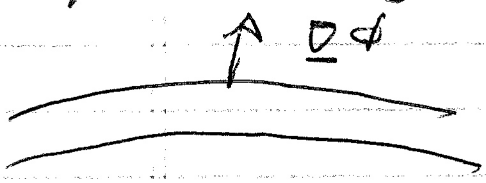
Now, to describe propagation:



i.e. each point on surface ϕ = const at t emits spherical disturbance.

- sum of spherical disturbances defines new constant phase surface. Curvature due $c(\underline{X})$.
  Envelope of spheres $\Rightarrow$ wavefront at $t_o+\Delta t$

- rays orthogonal to wave fronts.



Now, infinitesimal displacement vector along ray $\equiv d\underline{s}$

i.e. $d\underline{s} \parallel \underline{\nabla}\phi$

then, since equivalent to advance in space or time,

$$\underline{\nabla}\phi \cdot d\underline{s} = \omega\, dt$$

$$|\underline{\nabla}\phi|\,|d\underline{s}| = \omega\, dt$$

$dt = ds/c$ (by definition)

$$\Rightarrow \quad |\underline{\nabla}\phi|\,|d\underline{s}| = \omega \frac{ds}{c}$$

$\therefore$

$$|\underline{\nabla}\phi| = \omega/c$$

$$\Rightarrow \boxed{(\underline{\nabla}\phi)^2 = \omega^2/c^2}$$

– eikonal equation

reduces wave eqn to phase eqn.

$\Rightarrow$ eqn. for spatial evolution ϕ.

N.B. – Can obtain directly from Helmholtz Eqn.

$$\nabla^2\psi + \frac{\omega^2}{c(x)^2}\psi = 0$$

$$\psi = A\, e^{i\phi(\underline{x})/\epsilon}$$

$\epsilon \to 0$
(short wavelength)

$\Rightarrow$

$$\left[-\frac{(\underline{\nabla}\phi)^2}{\epsilon^2}A + i\frac{\underline{\nabla}^2\phi}{\epsilon}A + 2i\frac{\underline{\nabla}A\cdot\underline{\nabla}\phi}{\epsilon} + \nabla^2 A\right]e^{i\phi} = -\omega^2/c(x)^2\, A e^{i\phi}$$

so dominant balance

$$+\frac{(\nabla\phi)^2}{\epsilon^2} = \frac{\omega^2}{c(\underline{x})^2}$$

now absorb ϵ to ϕ.

- note eikonal lowers order of problem $\Rightarrow$ first order pde.

Now, by construction

$\underline{\nabla}\phi \cdot d\underline{\sigma} \equiv$ net phase increment along ray.

so $\underline{\nabla}\phi = \underline{k} = \underline{k}(\underline{x})$
$\qquad\qquad\downarrow$
in sense of WKB

(n.b. generally, $\partial\phi/\partial t = -\omega$).

$$\phi = \int \underline{k}\cdot d\underline{x} = \int \underline{\nabla}\phi \cdot d\underline{x}$$

$$= \int \underline{k}\cdot d\underline{\sigma}$$

$$\psi = A \exp\left[i\left(\int \underline{k}\cdot d\underline{x} - \omega t\right)\right]$$

is eikonal approximation to wave fctn.

N.B. ⇒ $\underline{k}$ specifies ray direction

⇒ Now, seek equations which evolve ray path in time, space i.e. give
- ray position $\underline{x}$ as fctn of
- ray direction $\underline{k}$

time.

⇒ defines mechanics problem.

a.) Poor Man's Version

- For linear wave, have $\omega = \text{const.}$

Since $\omega = \omega(\underline{k}, \underline{x})$ ⇒

$$\frac{d\omega}{dt} = 0 = \cancel{\frac{\partial \omega}{\partial t}} + \frac{\partial \omega}{\partial \underline{k}} \cdot \frac{d\underline{k}}{dt} + \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{d\underline{x}}{dt}$$

$$\Rightarrow \boxed{\begin{aligned} d\underline{k}/dt &= -\partial\omega/\partial \underline{x} \\ d\underline{x}/dt &= \partial\omega/\partial \underline{k} = \underline{v}_{gr} \end{aligned}}$$

eikonal equations

with, of course;

$$\omega^3 = c(\underline{x})^2 k^2$$

$$\cancel{2}\omega\, \partial\omega = \cancel{2}\underline{k}\cdot\partial\underline{k}\;\; c(\underline{x})^2$$

$$\underline{k} = k\,\hat{k}$$

$$\hat{k} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\partial\omega = \hat{k}\cdot\partial\underline{k}\;\; c(x)$$

$$d\omega/\partial\underline{k} = c(\underline{x})\,\hat{k}$$

$\equiv$ group velocity.

$$\frac{\partial\omega}{\partial\underline{x}} = \frac{\partial}{\partial\underline{x}}\left[c(x)^2 k^2\right]^{1/2} = k\,\frac{\partial c(\underline{x})}{\partial\underline{x}}$$

$\Rightarrow$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= c(\underline{x})\,\hat{k} \\ \frac{d\underline{k}}{dt} &= -k\,\frac{\partial c(\underline{x})}{\partial\underline{x}} \end{aligned}}$$

$c(\underline{x})$ profile determines ray path.

eikonal equation for acoustics.

b.) More Rigorously ....

$$\Phi = \int\left[\underline{k}\cdot d\underline{x} - \omega dt\right] \rightarrow \text{total phase}$$

$$dS = L\,dt = (p\dot{q} - H)\,dt$$

$$d\Phi = \underline{k}\cdot d\underline{x} - \omega\,dt = (\underline{k}\cdot\dot{\underline{x}} - \omega)\,dt$$

Now, assert ray will follow path which extremizes Φ, i.e. {minimizes accumulated phase?}.

Note: analogy of phase and action.

: later demonstrate connection to Fermat.

$$\delta\Phi = \delta\int\left[\underline{k}\cdot d\underline{x} - \omega\,dt\right] = 0$$

$$= \int\left\{\left[\delta\underline{k}\cdot d\underline{x} + \underline{k}\cdot\delta d\underline{x}\right] - \left(\frac{\partial\omega}{\partial\underline{k}}\cdot\delta\underline{k} + \frac{\partial\omega}{\partial\underline{x}}\cdot\delta\underline{x}\right)dt\right\}$$

as usual, $\delta\underline{x} = \delta\underline{k} = 0$, at end points.

So integrating by parts:

$$\delta\Phi = \int\left\{\left[\delta\underline{k}\cdot d\underline{x} - d\underline{k}\cdot\delta\underline{x}\right] + \text{e.p.} - \left[\left(\frac{\partial\omega}{\partial\underline{k}}\cdot\delta\underline{k}\right) + \left(\frac{\partial\omega}{\partial\underline{x}}\cdot\delta\underline{x}\right)\right]dt\right\}$$

$$\Rightarrow \quad d\underline{x} = \left(\frac{\partial \omega}{\partial \underline{k}}\right) dt$$

$$d\underline{k} = -\left(\frac{\partial \omega}{\partial \underline{x}}\right) dt$$

$$\Rightarrow \quad \boxed{\begin{aligned} \frac{d\underline{x}}{dt} &= \frac{\partial \omega}{\partial \underline{k}} \\ \frac{d\underline{k}}{dt} &= -\frac{\partial \omega}{\partial \underline{x}} \end{aligned}}$$

→ eikonal equations

Note:

→ evolve ray in $\underline{x}$, $\underline{k}$ phase space.

$\underline{x}$ → position

$\underline{k}$ → direction momentum

→ Hamiltonian equations for ray in $(\underline{x}, \underline{k})$ phase space.

i.e.
$$\frac{\partial}{\partial \underline{x}} \cdot \frac{d\underline{x}}{dt} + \frac{\partial}{\partial \underline{k}} \cdot \frac{d\underline{k}}{dt}$$

$$= \frac{\partial}{\partial \underline{x}} \cdot \frac{\partial \omega}{\partial \underline{k}} - \frac{\partial}{\partial \underline{k}} \cdot \frac{\partial \omega}{\partial \underline{x}} = 0.$$

→ since eikonal equations Hamiltonian, can define:

$\rho(\underline{x}, \underline{k}, t) \equiv$ wave density in $\underline{x}, \underline{k}$ phase space

$N(\underline{x}, \underline{k}, t)$

- wave action density
- ~ Wigner dist.
- ~ intensity.

and use Liouville's Thm:

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{v}_{gr} \cdot \frac{\partial \rho}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial \rho}{\partial \underline{k}} = 0}$$

- wave kinetic eqn.
- relates ρ, and intensity, to $c(\underline{x})$ profiles, for acoustics
- gives intensity evoln.
- applications in radiation hydro, quasi-particle evolution, etc.

→ Obvious analogy:

Particles	Rays
H	ω
$\underline{p}$	$\underline{k}$
$\underline{q}$	$\underline{x}$
S	ϕ